BARGAINING SETS OF COOPERATIVE GAMES WITHOUT SIDE PAYMENTS

BY

BEZALEL PELEG

ABSTRACT

In this paper an analogue of the bargaining set $M_1^{(i)}$ is defined for cooperative games without side payments. An existence theorem is proved for games of pairs, while it is shown by an example that no general existence theorem holds.

One of the important questions in the theory of bargaining sets is the question of existence of stable payoff configurations. In [3] and [5] it is proved that for every coalition structure $\mathbf{B}$, in an n-person game with transferable utilities, there exists a payoff vector x such that the individually rational payoff configuration $(x,\mathbf{B}) \in M_1^{(i)}$. In this paper we investigate the validity of the above theorem for cooperative games without side payments. We find that it is valid for games of pairs*, while for games with non-trivial coalitions which contain more than two players, it is not always true. It is still possible that basically different generalizations of $M_1^{(i)}$ will lead to existence theorems.

As this paper belongs both to the areas of bargaining sets and cooperative games without side payments, the reader is referred to introductory papers in both fields: [2] in the first field, and [1] in the second.

§1. Definitions. Let N be a finite set and let B be a subset of N. A *B-vector* x^B is a real function defined on B whose value at $i \in B$ *is* x^i. *The superscript* N is omitted. E^B denotes the euclidean space of all the vectors x^B. We write $x^B \geqq y^B$ if $x^i \geqq y^i$ for all $i \in B$; $x^B > y^B$ is interpreted similarly. We now give the definition of a cooperative game without side payments in characteristic function form:

Definition 1.1. An *n-person game* is a pair (N,v), where N is a set with n members, and v is a function that carries each subset B of N into a subset $v(B)$ of E^B so that (i) $v(B)$ is closed and convex; and (ii) if $x^B \in v(B)$ and $x^B \geqq y^B$ then $y^B \in v(B)$.

N is the *set of players* and its members will be denoted by the numbers $1,\dots,n$. v is the *characteristic function*; we assume that it satisfies $v(\{i\}) = \{x^i : x^i \leqq 0\}$ for all $i \in N$, and $v(B) \supset \times_{i\in B} v(\{i\})$, for all $B \subset N$.

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* Our method of proof is similar to those in a detailed version of [3], (to appear in Studies in Mathematical Economics, Essays in Honor of O. Morgenstern, M. Shubik ed.)

Let (N,v) be an n-person game. For $B \subset N$ we denote $\bar{v}(B) = \{x^B : x^B \in v(B), x^B \geqq 0$, there is no $y^B \in v(B)$ such that $y^B > x^B\}$. A *coalition structure* (c.s.) is a partitition of N.

DEFINITION 1.2. An *individually rational payoff configuration* (i.r.p.c.) is a pair $(x,\boldsymbol{B})$, where $\boldsymbol{B}$ is a c.s. and $x \in E^N$ satisfies: $x^B \in \bar{v}(B)$ for all $B \in \boldsymbol{B}$.

An i.r.p.c. represents a possible outcome of (N,v).

DEFINITION 1.3. Let $(x,\boldsymbol{B})$ be an i.r.p.c. and $i,j \in B \in \boldsymbol{B}$, $i \neq j$. An *objection* of i against j in $(x,\boldsymbol{B})$ is a Q-vector y^Q that satisfies: $i \in Q$, $j \notin Q$, $y^k > x^k$ for all $k \in Q$, and $y^Q \in \bar{v}(Q)$.

DEFINITION 1.4. Let $(x,\boldsymbol{B})$ be an i.r.p.c. and y^Q an objection of player i against player j in $(x,\boldsymbol{B})$. A *counter objection* of j against i is an R-vector z^R that satisfies: $j \in R$, $i \notin R$, $z^k \geqq x^k$ for all $k \in R$, $z^k \geqq y^k$ for $k \in R \cap Q$, and $z^R \in \bar{v}(R)$.

An i.r.p.c. $(x,\boldsymbol{B})$ is *stable* if for each objection in $(x,\boldsymbol{B})$ there is a counter objection. The set of all stable i.r.p.c.'s is called the *bargaining set* $\tilde{\mathcal{M}}_1^{(i)}$.

Let $(x,\boldsymbol{B})$ be an i.r.p.c. An objection in $(x,\boldsymbol{B})$ is *justified* if it cannot be countered. Let $i,j \in B \in \boldsymbol{B}$, $i \neq j$. We write $i \gtrsim j$ in $(x,\boldsymbol{B})$, if j has no justified objection against i in $(x,\boldsymbol{B})$. We also denote by $X(\boldsymbol{B})$ the set of all the payoff vectors y such that $(y,\boldsymbol{B})$ is an i.r.p.c., and by E_i the set $\{y : y \in X(\boldsymbol{B}), i \gtrsim k$ in $(y,\boldsymbol{B})$ for all $k \in B - \{i\}\}$.

If $B \subset N$ we denote by $|B|$ the number of members of B. An n-person game (N,v) is a *game of pairs* if $v(B) = \times_{i \in B} v(\{i\})$ whenever $B \subset N$ and $|B| \neq 2$.

§2. Existence theorem for the bargaining set $\tilde{\mathcal{M}}_1^{(i)}$ of games of pairs.

Let (N,v) be a game of pairs. We remark that if $B \subset N$ then $\bar{v}(B)$ is homeomorphic to a closed interval; so if $\boldsymbol{B}$ is a c.s. then $X(\boldsymbol{B})$ is homeomorphic to a cartesian product of closed intervals.

The following lemma is not difficult to prove

LEMMA 2.1. *Let $B = \{i,j\}$ be a subset of N; the function*

$$x^i(x^j) = \max\{y^i : y^B \in \bar{v}(B), y^j = x^j\}$$

is defined and continuous for

$$0 \leqq x^j \leqq \max\{y^j : y^B \in \bar{v}(B)\}.$$

LEMMA 2.2. *Let $\boldsymbol{B}$ be a c.s. and $i \in B \in \boldsymbol{B}$; then E_i is a closed subset of $X(\boldsymbol{B})$.*

Proof. If $|B| \neq 2$ then* $E_i = X(\boldsymbol{B})$; so only the case $|B| = 2$ is left. Without loss of generality $B = \{1,2\}$ and $i = 1$. We shall prove that E_1 is closed by showing that $X(\boldsymbol{B}) - E_1$ is open relative to $X(\boldsymbol{B})$. Let $x_0 \in X(\boldsymbol{B}) - E_1$. 2 has a justified

* Since $x^i = 0$ implies that $x \in E_i$.

objection y^Q against 1 in $(x_0, \boldsymbol{B})$. Without loss of generality $Q=\{2,3\}$. Since 1 has no counter objection to y^Q we must have:

(a) $x_0^1 > 0$;

(b) either $x_0^1 > \max\{x^1 : x^{\{1,j\}} \in \bar{v}(\{1,j\})\}$ or $x_0^j > x^j(x_0^1)$, for all $j \in N-\{1,2,3\}$;

(c) either $x_0^1 > \max\{x^1 : x^{\{1,3\}} \in \bar{v}(\{1,3\})\}$ or $y^3 > x^3(x_0^1)$.

Since all the functions of x_0 that appear in (a), (b) and (c) are continuous, we can find a set F, open in $X(\boldsymbol{B})$, that contains x_0, and such that if $z \in F$ then (a), (b) and (c) are satisfied with z in place of x_0 and also $y^2 > z^2$ and $y^3 > z^3$. So y^Q is a justified objection of 2 against 1 in $(z, \boldsymbol{B})$: it follows that $F \subset X(\boldsymbol{B}) - E_1$, which shows that $X(\boldsymbol{B}) - E_1$ is open relative to $X(\boldsymbol{B})$.

Let $\boldsymbol{B}$ be a c.s. and $B \in \boldsymbol{B}$. We denote $U_B = \cap_{i \in B} E_i$. Also, if $x \in X(\boldsymbol{B})$, we denote $V_B(x^{N-B}) = \{y^B : (y^B, x^{N-B}) \in U_B\}$.

LEMMA 2.3. *Let $(x, \boldsymbol{B})$ be an i.r.p.c. If $B \in \boldsymbol{B}$ then $V_B(x^{N-B})$ is homeomorphic to a closed interval.*

Proof.* If $|B| \neq 2$ then $V_B(x^{N-B})$ consists of one point; so only the case $|B| = 2$ is left. Without loss of generality $B = \{1, 2\}$. We denote $G_i = \{y^B : (y^B, x^{N-B}) \in E_i\}$, $i = 1,2$. G_1 and G_2 are non-void closed subsets of $\bar{v}(B)$ and $G_1 \cap G_2 = V_B(x^{N-B})$. If a point $y^B \in G_i$ then every point $z^B \in \bar{v}(B)$ that satisfies $z^i \leqq y^i$ is also in G_i. So to prove that $G_1 \cap G_2$ is homeomorphic to a closed interval it is sufficient to show that $G_1 \cap G_2 \neq \emptyset$. Since $\bar{v}(B)$ is connected we shall complete the proof if we shall show that $\bar{v}(B) = G_1 \cup G_2$.

Assume that $y^B \in \bar{v}(B) - (G_1 \cup G_2)$. 1 has a justified objection z_1^Q against 2 in $((y^B, x^{N-B}), \boldsymbol{B})$ and 2 has a justified objection z_2^R against 1 in the same i.r.p.c. We have $|R| = |Q| = 2$. If $R \cap Q = \emptyset$ then z_1^R is a counter objection to z_2^R. If $R \cap Q \neq \emptyset$ then it contains a single player j. In this case if $z_1^j \geqq z_2^j$ then z_1^Q is a counter objection to z_2^R, and if $z_2^j > z_1^j$ then z_2^R is a counter objection to z_1^Q. So the assumption $\bar{v}(B) - (G_1 \cup G_2) \neq \emptyset$ leads to a contradiction and the proof is completed.

THEOREM 2.4. *Let $\boldsymbol{B}$ be a c.s. in a game of pairs; then there always exists a payoff vector x such that the i.r.p.c. $(x, \boldsymbol{B}) \in \tilde{M}_1^{(i)}$.*

Proof. For $x \in X(\boldsymbol{B})$ let $T(x) = \underset{B \in \boldsymbol{B}}{\times} V_B(x^{N-B})$. Since the sets U_B, $B \in \boldsymbol{B}$, are closed, T is upper semi-continuous. Lemma 2.3 implies that for each $x \in X(\boldsymbol{B})$ $T(x)$ is homeomorphic to a cartesian product of closed intervals. By the fixed-point theorems of Eilenberg and Montgomery [4] T has a fixed point, i.e. there is a payoff vector $x_0 \in X(\boldsymbol{B})$ such that $x_0 \in T(x_0)$. From the definiton of T it is clear that $(x_0, \boldsymbol{B}) \in \tilde{M}_1^{(i)}$.

We now give an example which shows that Theorem 2.4 cannot be generalised to games with non-trivial coalitions which contain more than two players.

* See [2] Lemma 7.2. for the proof for games with side payments where $\bar{v}(B)$ is an interval

EXAMPLE 2.5. Let (N, v) be a 4-person game given by: $v(\{1, 2\}) = \{x^{\{1,2\}} : x^1 + x^2 \leqq 1\}$, $v(\{1,3,4\}) = \{x^{\{1,3,4\}} : x^1 \leqq 2,\ x^3 \leqq 3,\ x^4 \leqq 4\}$, $v(\{2,3,4\}) = \{x^{\{2,3,4\}} : x^2 \leqq 2, x^3 \leqq 4, x^4 \leqq 3\}$, and $v(B) = \times_{i \in B} v(\{i\})$ for the remaining coalitions $B \subset \{1,2,3,4\}$. If $\mathcal{B} = \{\{1,2\}, \{3\}, \{4\}\}$ then, as the reader can easily verify, there is no $x \in X(\mathcal{B})$ such that $(x, \mathcal{B}) \in \tilde{M}_1^{(i)}$.

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